“One-ended spanning subforests and treeability of groups”

by Conley–Gaboriau–Marks–Tucker-Drob

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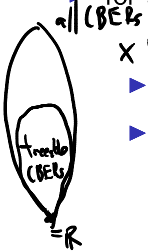
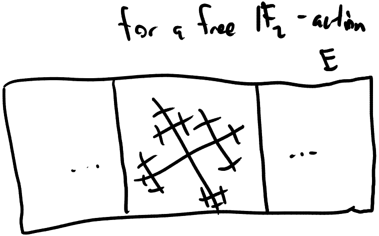
Preliminaries

- **countable Borel equivalence relation (CBER)** E (or $\mathcal{R}$) $\subseteq X^2$:
  $E \subseteq X^2$ Borel, $[x]_E$ ctbl $\forall x \in X$.
- for Borel $\Gamma \curvearrowright X$, **orbit equivalence relation** E_Γ (or $\mathcal{R}_\Gamma$)
  $x \, E_\Gamma \, y :\Leftrightarrow \exists \gamma \in \Gamma$ s.t. $\gamma \cdot x = y$
- for a **Borel graph** $G \subseteq X^2$, E_G (or $\mathcal{R}_G$)
  $G \cap \Delta_X = \emptyset$, $G = G^{-1}$
  $x \, E_G \, y :\Leftrightarrow \exists\, x = x_0 \, G \, x_1 \, G \dots G \, x_n = y$
  - G is a **graphing** of E_G
  - G is a **treeing** if furthermore, G is acyclic

  Example If $\Gamma \curvearrowright X$ is free and $\Gamma = \langle S \rangle$,
  $x \, G \, y :\Leftrightarrow \exists \gamma \in S$ s.t. $\gamma \cdot x = y$.





- for a Borel prob measure μ on X, a property of E holds μ-a.e. if it holds on an E-inv μ-conull set.

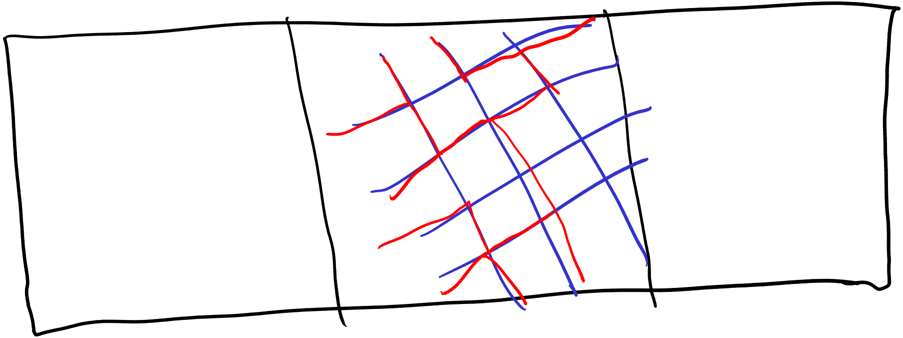
Planar graphs are measure treeable

Theorem (CGMT 2021)

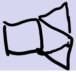
"each component is planar"

Let $G \subseteq X$ be a locally finite Borel planar graph.*
Then for any Borel probability measure μ on X, G has a Borel subtreeing μ-a.e.
In particular, E_G is treeable μ-a.e.

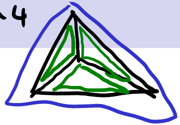
∃ Borel acyclic $T \subseteq G$ s.t. $E_T = E_G$.



Planar graphs



K4



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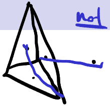


A graph G on vertices X is **planar** if $\exists$ a **planar embedding** $f : X \hookrightarrow \mathbb{R}^2$, &

$f_e : [0,1] \hookrightarrow \mathbb{R}^2$ for each $e=(x,y)\in G$, which have disj. images except at endpts.
$f_e(0)=x$, $f_e(1)=y$

A **facial cycle** $(e_1, \dots, e_n)$ (edges) is a cycle whose image under f is $\partial \bar{K}$ for a bounded connected comp K of $\mathbb{R}^2 \setminus \text{im}(f)$

- each edge belongs to at most 2 facial cycles
- the char fns of facial cycles are lin indep over $\mathbb{Z}/2\mathbb{Z}$
- (if X is finite) they span the char fns of all cycles over $\mathbb{Z}/2\mathbb{Z}$

A **2-basis** in G is a family of cycles obeying these 3 conditions.

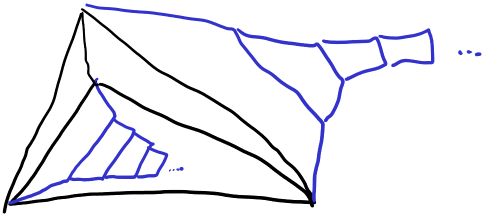
Theorem (Mac Lane 1937)

If X is finite, each 2-basis is the set of facial cycles for some planar embedding.

Planar graphs

An **accumulation point** of a planar embedding $f : (X, G) \to \mathbb{R}^2$ is a pt which is a limit of an inf seq of distinct vertices or edges.

Example

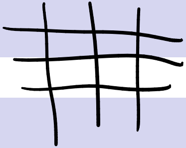


Theorem (Thomassen 1980)

- If a planar emb has no accumulation pts, the facial cycles form a 2-basis.
- If G is loc fin, each 2-basis is given by a planar embedding.

A Borel graph $G \subseteq X^2$ is **Borel planar** if it has a Borel 2-basis.
as a subset of $[X]^{<\omega}$.

Planar groups are measure treeable



Theorem (CGMT 2021)

Let $G \subseteq X$ be a locally finite Borel planar graph.
Then for any Borel probability measure μ on X, G has a Borel subtreeing μ-a.e.
In particular, E_G is treeable μ-a.e.

Corollary (CGMT 2021)

B s.t. for each $(e_1, \ldots, e_n) \in B$, $(\gamma \cdot e_1, \ldots, \gamma \cdot e_n) \in B$

If Γ acts freely on a connected planar graph ~~with equivariant 2-basis~~, then every free Borel Γ-action is treeable μ-a.e. for every Borel probability measure μ.

Γ is measure-strongly-treeable $\Rightarrow$ Γ strongly-treeable $\Rightarrow$ Γ treeable

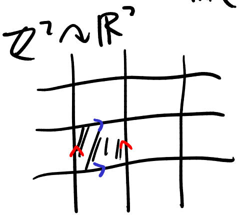
(strongly-treeable:) every pmp action is treeable

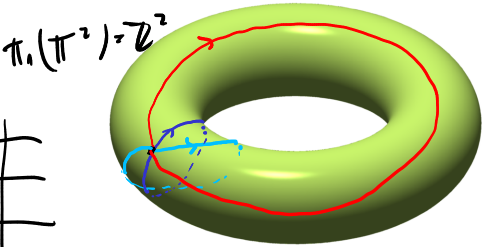
(treeable:) $\exists$ a pmp treeable action

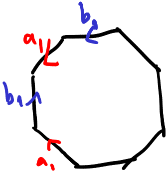
Surface groups

Let Σ be a closed orientable surface.
Its **fundamental group** $\pi_1(\Sigma)$ is {loops} / cts deformation.

$\langle a_1, b_1, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots \rangle$







Every surface Σ (except S^2) is a free quotient $\widetilde{\Sigma}/\pi_1(\Sigma)$ where $\widetilde{\Sigma} \cong \mathbb{R}^2$.

Corollary (CGMT 2021)

Every free Borel action of $\pi_1(\Sigma)$ is treeable μ-a.e.

Some other treeable groups

Theorem (CGMT 2021)

Let $G \subseteq X$ be a locally finite Borel planar graph.
Then for any Borel probability measure μ on X, G has a Borel subtreeing μ-a.e.
In particular, E_G is treeable μ-a.e.

A group Γ is **elementarily free** if it is elementarily equivalent to $\mathbb{F}_2$.

Corollary (CGMT 2021)

Every free Borel action of a f.g. elementarily free group is treeable μ-a.e.

Proof uses an explicit construction of a space X with $\pi_1(X) \cong \Gamma$
(Sela 2006, Guirardel–Levitt–Sklinos 2020).

Some other treeable groups

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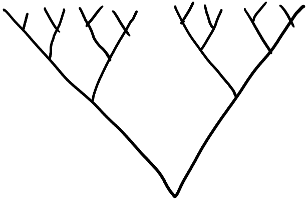
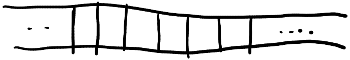
Corollary (CGMT 2021)

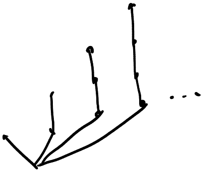
Every free Borel action of $\mathrm{Isom}(\mathbb{H}^2)$ *is treeable μ-a.e.*

Ends of graphs

Let G be a graph on X. An **end** in G is "







For each finite $F \subseteq X$, look at $\pi_0(G|(X \setminus F)) :=$

For finite $F_0 \subseteq F_1 \subseteq X$,

The **space of ends** of (X, G) is $\partial G :=$

If G is locally finite:

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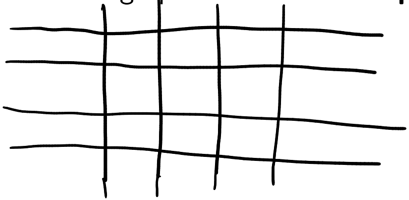
One-ended spanning subforests

Let $G \subseteq X^2$ be a Borel graph. A **one-ended spanning subforest** is



Conjecture (CGMT 2021)

Let $G \subseteq X^2$ be a locally finite Borel graph with E_G μ-a.e. nonsmooth. TFAE:

(i) *G is μ-a.e. not 2-ended.*

(ii) *G has a Borel one-ended spanning subforest μ-a.e.*

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- (CMT 2016)
- (CGMT 2021)

Cutting cycles along a one-ended subforest

Theorem (CGMT 2021)

Let $G \subseteq X$ be a locally finite Borel planar graph.
Then for any Borel probability measure μ on X, G has a Borel subtreeing μ-a.e.

Proof idea:

Cutting cells in higher-dimensional complexes

Corollary (CGMT 2021)

For a compact surface Σ, every free Borel action of $\pi_1(\Sigma)$ is treeable μ-a.e.

Theorem (CGMT 2021)

For a compact aspherical n-manifold M, every free Borel action of $\pi_1(M)$ admits a "Borel family of contractible $(n-1)$-dim'l simplicial complexes on each class", up to μ-a.e. Borel reducibility.